Covariant Schwinger Terms

C. Adam, C. Ekstrand, T. Sýkora

Abstract

There exist two versions of the covariant Schwinger term in the literature. They only differ by a sign. However, we shall show that this is an essential difference. We shall carefully (taking all signs into account) review the existing quantum field theoretical computations for the covariant Schwinger term in order to determine the correct expression.

1 Introduction

One essential feature of chiral gauge theories is the violation of gauge invariance when chiral (Weyl) fermions are quantized. This loss of gauge invariance results in a non-invariance of the vacuum functional in an external gauge field under gauge transformations and in a non-conservation (anomalous divergence) of the corresponding gauge current in a Lagrangian (or space-time) formulation ("anomaly" [1]–[3]), or in anomalous contributions to the equal-time commutators of the generators of time-independent gauge transformations in a Hamiltonian formulation ("Schwinger term" or "commutator anomaly", [4]–[9]). One regularization scheme, where all the anomalous terms are related to functional derivatives of the vacuum functional, leads to the so-called consistent anomalies. These anomalies have to obey certain consistency conditions because of their relation to functional derivatives [10]. One way of determining these consistent anomalies (up to an overall constant) is provided by the descent equations of Stora and Zumino [11, 12]. They provide a simple algebraic scheme – based on some geometrical considerations – for the computation of the consistent chiral anomaly in the space-time formalism and for the corresponding equal-time commutator anomaly, the Schwinger term (as well as for higher cochain terms).

On the other hand, it is possible to choose a gauge-covariant regularization for the gauge current. This covariant current cannot be related to a functional derivative of the vacuum functional (because of the gauge non-invariance of the latter). As a consequence, the covariant current anomaly does not obey the consistency condition. Nevertheless, there exists a covariant counterpart for each consistent cochain in the descent equations. The first derivation of an algebraic computational scheme for covariant cochains appears to be the one by Tsutsui [13], using the anti-BRST formalism. Further, a covariant version of the descent equations was derived by Kelnhofer in [14]. The covariant cochains resulting from the calculations by Tsutsui on one hand, and by Kelnhofer on the other hand are, in

fact, different, as was shown in [15]. Tsutsui's and Kelnhofer's formulas predict the same anomaly in space-time, but their Schwinger terms differ by a sign. The higher cochains (with more than 2 ghosts) seem to be unrelated. We shall give an answer to which of the two formulas is correct, in the sense that it is reproduced by a full quantum field theoretic calculation.

The easiest way to do this calculation would be to compute one of the higher covariant anomalies in some quantum field theoretic setting, because these higher covariant anomalies are given by completely different expressions in [13] and [14]. However, although it has been claimed that the higher cochains can have a physical meaning, this is far from understood. It is therefore not sound to use these terms to argue which of the two formulas is correct. Instead, we shall use the sign of the Schwinger term as a referee.

We shall use three methods to determine the correct expression for the covariant Schwinger term. They all have to be used with care since we are after the sign difference between Tsutsui's and Kelnhofer's predictions. The first two methods are to apply the quantum field theoretical calculation schemes that have been used by Adam [17] and by Hosono and Seo [18], respectively.

The third method is the one by Wess [19], relating the Schwinger term (consistent or covariant) in any even dimensional space-time with the corresponding space-time anomaly. This method was used by Schwiebert [20] for the consistent case and by Kelnhofer [21] in the covariant case. Thus, by using the expression for the covariant anomaly (which everyone agrees on) the covariant Schwinger term can be determined. Again, care has to be taken. For this, we first perform the calculation in the consistent formalism and set conventions so the result agrees with what is predicted by the descent equations. The corresponding covariant computation is then to determine the covariant Schwinger term including the sign. We shall perform these calculation in 1+1 and 3+1 dimensions.

From the explicit calculations we find that our quantum field theoretical methods produce the same expression (i.e. sign) as Kelnhofer's covariant descent equations. This result is not obvious since Kelnhofer's approach, as well as Tsutsui's, seems to be based only on the requirement of covariance. We shall however show that there is in fact a natural interpretation of the Kelnhofer formula, as one would expect.

Our paper is organized as follows. In Section 2 we briefly describe the geometrical setting for the description of anomalies and review the derivation of the consistent and covariant chain terms. In the covariant case, both Tsutsui's and Kelnhofer's versions of the chain terms are given and a geometrical description of Kelnhofer's construction is provided. In Section 3 the consistent and covariant Schwinger terms in 1+1 dimensions are calculated using the methods of [17] and of [18]. Finally, in Section 4 the Schwinger terms in 1+1 as well as in 3+1 dimensions are calculated with the help of the method of Wess [19].

2 Consistent and covariant cochains

We shall start with deriving the consistent chiral anomaly for a non-abelian gauge theory. Consider therefore Weyl fermions ψ coupled to an external gauge field $A \in \mathcal{A}$. $\mathcal{A}$ is the affine space of of gauge connections and the gauge group G is assumed to be a compact, semi-simple matrix group. We assume that the space-time M is a smooth, compact, oriented, even-dimensional and flat Riemannian spin manifold without boundary. The group $\mathcal{G}$ of gauge transformations consists of diffeomorphisms of a principal bundle $P \xrightarrow{G} M$ such that the base remains unchanged. It acts on $\mathcal{A}$ by pull-back and to make this action free we restrict to gauge transformations that leaves a reference point $p_0 \in P$ fixed.

The generating functional is given by

$$\exp(-W(A)) = \int_{\psi,\bar{\psi}} \exp(-\int_M \bar{\psi} \partial\!\!\!/_A^+ \psi d^{2n}x), \tag{1}$$

where W is the effective action and $\partial\!\!\!/_A^+ = \partial\!\!\!/_A(1+\gamma_5)/2 = \gamma^\mu(\partial_\mu + A_\mu)(1+\gamma_5)/2$. We shall use conventions such that γ^μ is hermitean and A_μ is anti-hermitean. It has been argued that a correct interpretation of the generating functional is as a section of the determinant line bundle $\mathrm{DET} i\partial\!\!\!/_A = \det \ker i\partial\!\!\!/_A^+ \otimes (\det \mathrm{coker} i\partial\!\!\!/_A^+)^*$. It can be viewed as a functional by comparing with some reference section. Associated with the determinant line bundle is a connection with corresponding curvature

$$F = -2\pi i \frac{1}{(n+1)!} \left(\frac{i}{2\pi}\right)^{n+1} \int_M \mathrm{tr}\left(\mathcal{F}^{n+1}\right), \tag{2}$$

[22]. Here, $\mathcal{F} = (d+\delta)(A + (d_A^* d_A)^{-1} d_A^*) + (A + (d_A^* d_A)^{-1} d_A^*)^2$ is a curvature of the principal bundle $P \times \mathcal{A} \to M \times \mathcal{A}$ and δ is the exterior differential in $\mathcal{A}$. The choice of $\mathcal{F}$ is motivated by gauge invariance of the determinant line bundle, [23, 24].

Recall that

$$\mathrm{tr}\left(\mathcal{F}_2^{n+1}\right) - \mathrm{tr}\left(\mathcal{F}_1^{n+1}\right) = (d+\delta)\omega_{2n+1}(\alpha_2, \alpha_1) \tag{3}$$

for

$$\omega_{2n+1}(\alpha_2, \alpha_1) = (n+1) \int_0^1 dt \mathrm{tr}\left((\alpha_2 - \alpha_1)\mathcal{F}_t^n\right) \tag{4}$$

and $\mathcal{F}_t$ the curvature of $(1-t)\alpha_1 + t\alpha_2$, holds for any connections α_1, α_2 with curvatures $\mathcal{F}_1, \mathcal{F}_2$. Using this in eq. (2) gives the following expression for the connection of the determinant line bundle:

$$-2\pi i \frac{1}{(n+1)!} \left(\frac{i}{2\pi}\right)^{n+1} \int_M \omega_{2n+1}(A + (d_A^* d_A)^{-1} d_A^*, 0). \tag{5}$$

The (infinitesimal) consistent anomaly is the variation of the effective action under gauge transformations. Thus, it is the negative of the restriction of (5) to gauge directions, i.e. the fibre directions of $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. Along such directions, δ becomes the BRST operator and $(d_A^* d_A)^{-1} d_A^*$ becomes the ghost v. Thus, the consistent anomaly is $c_n \int_M \omega_{2n+1}(A+v, 0)$, with $c_n = -\frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^n$. Since M is $2n$-dimensional, it is only the term with one ghost

in the expansion of $\omega_{2n+1}(A+v,0)$ that will give a contribution to the anomaly. We let $\omega^k_{2n+1-k}(A+v,0)$ denote the part of $\omega_{2n+1}(A+v,0)$ that contains k number of ghosts. Then $c_n\omega^1_{2n}(A+v,0)$ is the non-integrated anomaly. It is well-known, and explicitly proven in [25], that $c_n\int_M \omega^2_{2n-1}(A+v,0)$ is the Schwinger term. In this case M is to be interpreted as the odd-dimensional physical space at a fixed time. The forms $\omega^k_{2n+1-k}(A+v,0)$ can be computed by use of eq. (4). In 1+1 and 3+1 dimensions it gives the following result for the consistent anomaly and Schwinger term:

$$\begin{aligned} c_1\omega^2_1(A+v,0) &= c_1\mathrm{tr}(vdA) \\ c_1\omega^1_2(A+v,0) &= -c_1\mathrm{tr}\left(v^2A\right) \\ c_2\omega^4_1(A+v,0) &= c_2\mathrm{tr}\left(vd\left(AdA+A^3/2\right)\right) \\ c_2\omega^3_2(A+v,0) &= -c_2\mathrm{tr}\left(\left(v^2A+vAv+Av^2\right)dA+v^2A^3\right)/2. \end{aligned} \tag{6}$$

If the freedom is used to change the forms $\omega^k_{2n+1-k}(A+v,0)$ by cohomologically trivial terms (i.e. by coboundaries), then these forms can be given by the following compact expressions that were first derived by Zumino in [26]:

$$\begin{aligned} &\omega^k_{2n+1-k}(A+v,0) \\ \sim\ &(n+1)\begin{pmatrix} n \\ k \end{pmatrix}\int_0^1 dt\ (1-t)^k \mathrm{str}\left((dv)^k, A, \left(tdA+t^2A^2\right)^{n-k}\right) \end{aligned} \tag{7}$$

when $0 \le k \le n$ and

$$\begin{aligned} \omega^k_{2n+1-k}(A+v,0) \sim\ &(-1)^{k-n-1}\begin{pmatrix} n \\ k-n-1 \end{pmatrix}\left(\begin{pmatrix} k \\ k-n-1 \end{pmatrix}\right)^{-1} \\ &\times \mathrm{str}\left(v, (v^2)^{k-n-1}, (dv)^{2n-k+1}\right) \end{aligned} \tag{8}$$

when $n+1 \le k \le 2n+1$. Here str means the symmetrized trace and $\sim$ means equality up to a coboundary.

Above, we used eq. (3) for $\alpha_2 = A + (d^*_A d_A)^{-1} d^*_A$ and $\alpha_1 = 0$. When P is a non-trivial bundle it is no longer possible to let α_1 be zero. Instead, we let it be some fixed connection A_0 on P (which can be identified with a connection on $P \times \mathcal{A}$). By dimensional reasons, this does not change eq. (2). The consistent anomaly and Schwinger term with such a background connection can be computed in a similar way as above, one just uses $\omega^k_{2n+1-k}(A+v,A_0)$ instead. Since the expressions corresponding to eq. (7) and eq. (8) are long and not particularly illuminating we shall not present them here (parts of it can be found in [15]). The ideas behind the background connection are completely analogous with the case without a background. For example, they are consistent, but not gauge covariant. To obtain covariance, we choose as a background the field itself. We are then interested in the (non-consistent) terms coming from the expansion of $\omega^k_{2n+1-k}(A+v,A)$ in various ghost degrees. With use of eq. (4), the following expression was obtained in

[15]:

$$\omega^k_{2n+1-k}(A+v,A) \;=\; \sum_{j=0}^{[(k-1)/2]} \frac{n+1}{k-j}\begin{pmatrix} n-j \\ k-2j-1\end{pmatrix}\begin{pmatrix} n \\ j\end{pmatrix}\left(\begin{pmatrix} k \\ j\end{pmatrix}\right)^{-1}$$
$$\times\mathrm{str}\left(v,(\delta v)^j,(\delta A)^{k-2j-1},F^{n-k+j+1}\right), \qquad (9)$$

where a negative power on a factor means that the corresponding term is absent in the sum. Recall that $[(k-1)/2]$ is $(k-1)/2$ if k is odd and $(k-2)/2$ if k is even. The terms

$$\begin{aligned} c_n\omega^1_{2n}(A+v,A) &= c_n(n+1)\mathrm{tr}(vF^n) \\ c_n\omega^2_{2n-1}(A+v,A) &= c_n\frac{n(n+1)}{2}\mathrm{str}(v,\delta A,F^{n-1}) \end{aligned} \qquad (10)$$

are the non-integrated covariant anomaly and Schwinger term.

Let us summarize the results so far in the case of 1+1 and 3+1 dimensions in Tables 1 and 2, respectively (we use eq. (7) and (8) for the consistent formalism)

$n=1$	Anomaly	Schwinger term
Consistent	$c_1\int_M \mathrm{tr}((dv)A)$	$c_1\int_M \mathrm{tr}(vdv)$
Covariant	$c_1\cdot 2\int_M \mathrm{tr}\left(v\left(dA+A^2\right)\right)$	$-c_1\int_M \mathrm{str}(v(dv+2vA))$

Table 1

$n=2$	Anomaly	Schwinger term
Consistent	$c_2\int_M \mathrm{tr}((dv)AdA$ $+\frac{1}{2}(dv)A^3)$	$c_2\int_M \mathrm{tr}\left((dv)^2A\right)$
Covariant	$c_2\cdot 3\int_M \mathrm{tr}$ $\left(v\left(dA+A^2\right)^2\right)$	$c_2\cdot(-3)\int_M \mathrm{str}(v(dv+vA$ $+Av)(dA+A^2))$

Table 2

We shall now evaluate these forms on (anti-hermitean) infinitesimal gauge transformations $X,Y\in \mathrm{Lie}\mathcal{G}$. Let us do this explicitely for the consistent Schwinger term when $n=2$:

$$c_2\int_M \mathrm{tr}\left((dv)^2A\right)(X,Y) = -c_2\int_M \mathrm{tr}(\partial_i v\wedge\partial_j vA_k)\epsilon^{ijk}d^3x(X,Y)$$
$$= \; -c_2\int_M \mathrm{tr}((\partial_i X\partial_j Y-\partial_i Y\partial_j X)A_k)\epsilon^{ijk}d^3x. \qquad (11)$$

The corresponding evaluation of the other forms for $n = 1$ and $n = 2$ is listed in Tables 3 and 4, respectively.

$n = 1$	Anomaly	Schwinger term
Consistent	$-c_1 \mathrm{tr}((\partial_\mu X)A_\nu)\epsilon^{\mu\nu}$	$-2c_1 \mathrm{tr}\,(X\partial_x Y)$
Covariant	$2c_1 \mathrm{tr}(X(\partial_\mu A_\nu + A_\mu A_\nu))\epsilon^{\mu\nu}$	$2c_1 \mathrm{tr}(X\partial_x Y - [X,Y]A_x)$

Table 3

$n = 2$	Anomaly	Schwinger term
Consistent	$-c_2 \mathrm{tr}((\partial_\mu X)(A_\nu \partial_\rho A_\lambda + \frac{1}{2} A_\nu A_\rho A_\lambda))\epsilon^{\mu\nu\rho\lambda}$	$-c_2 \mathrm{tr}((\partial_i X \partial_j Y - \partial_i Y \partial_j X)A_k)\epsilon^{ijk}$
Covariant	$3c_2 \mathrm{tr}(X(\partial_\mu A_\nu + A_\mu A_\nu))(\partial_\rho A_\lambda + A_\rho A_\lambda)\epsilon^{\mu\nu\rho\lambda}$	$3c_2 \mathrm{tr}((X\partial_i Y - Y\partial_i X - -[X,Y]\,A_i + XA_iY - YA_iX)(\partial_j A_k + A_j A_k))\epsilon^{ijk}$

Table 4

That the covariant anomaly and Schwinger term can be computed by expansion of $\omega_{2n+1}(A+v, A)$ was discovered by Kelnhofer [14]. An alternative computational scheme leading to covariant cochains differing from the ones of Kelnhofer was given by Tsutsui [13]. To review his approach we reconsider eq. (4) for $\alpha_2 = A + v$ and $\alpha_1 = 0$. We can then view ω_{2n+1} as a function $\omega_{2n+1}(A+v|\mathcal{F})$ of $A+v$ and $\mathcal{F} = (d+\delta)(A+v) + (A+v)^2$:

$$\omega_{2n+1}(A+v|\mathcal{F}) = (n+1)\int_0^1 dt \mathrm{tr}\,((A+v)\mathcal{F}_t^n)\,, \quad \mathcal{F}_t = t\mathcal{F} + (t^2 - t)(A+v). \tag{12}$$

The covariance is broken by the operator δ in the expression for $\mathcal{F}$. Thus, $\omega_{2n+1}(A+v|\mathcal{F}')$, with $\mathcal{F}' = d(A+v)+(A+v)^2$, produces covariant terms. This is exactly the same terms as the ones appearing in Tsutsui's anti-BRST approach [15, 27]. In [15] (see [16] for $k = 2$) the following formula was given for the terms with a given ghost degree:

$$\omega_{2n+1-k}^k(A+v|\mathcal{F}') = \frac{n+1}{k} \mathrm{tr}\,(v(F - \delta(A+v))^n)_k\,, \tag{13}$$

where the index k on the right hand side means the part of the expression that has k number of ghosts. Comparison with eq. (9) reveals that this formula gives the same covariant anomaly but the covariant Schwinger term differs by a sign. The higher terms seem to be unrelated. This brings us to the question of who is right: Kelnhofer or Tsutsui?

The formula of Tsutsui seems to be motivated by nothing else than covariance. Kelnhofer's formula, on the other hand, seems to appear in a natural way: it is obtained by putting the background field equal to the field under consideration. In the computation of the Schwinger term from determinant line bundles for manifolds with boundary, one extends space to a cylindrical space-time, [25]. On one side of the cylinder one computes the Schwinger term by comparison of a fixed vacuum bundle (with respect to a background connection) on the other side of the cylinder. In this approach it is certainly possible to put the background field equal to the field itself, see [24] for details. This clearly defines a covariant Schwinger term in a natural way, suggesting that Kelnhofer's approach is the correct one. This geometrical approach would not have been possible with Tsutsui's result. This explains the importance of the sign of the covariant Schwinger term. In the forthcoming sections we shall demonstrate that indeed Kelnhofer's result for the covariant Schwinger term is reproduced by quantum field theoretic computations.

3 Calculations in 1+1 dimensions

3.1 Calculation of Adam

In this section we want to briefly review the calculation of the consistent and covariant Schwinger term that was performed in [17] for the Abelian case (the chiral Schwinger model). The generalization to the non-Abelian case is straight-forward and shall be displayed below, as well. In [17] the Hamiltonian formulation was used (therefore space-time is 1+1 dimensional Minkowski space), and the computation started from the second-quantized chiral fermion field operator in the interaction picture. For fermionic field operators the Dirac vacuum has to be introduced and operator products have to be normal-ordered w.r.t. the Dirac vacuum. For the introduction of the Dirac vacuum the Hilbert space of fermionic states is split into a positive and negative momentum sub-space (for chiral fermions in two dimensions energy equals momentum). For the negative momentum sub-space the role of creation and annihilation operators is then exchanged. At this point there are two possibilities to split. Either one may split w.r.t. eigenvalues of the free momentum operator $-i\partial_{x^1}$ and perform normal-ordering (denoted by N) for this Dirac vacuum. A well-known consequence of this normal-ordering is the fact that the current commutators acquire a central extension (Schwinger term). For a fermion of positive chirality (where the current obeys $J^0 = J^1 =: J$), the Schwinger term is

$$[NJ(x^0, x^1), NJ(x^0, y^1)] = -\frac{i}{2\pi}\delta'(x^1 - y^1) \tag{14}$$

(here the prime denotes derivative w.r.t. the argument). The second possibility is to split w.r.t. eigenvalues of the kinetic momentum operator $-i\partial_{x^1} + eA_1$. Again, a corresponding Dirac vacuum and normal ordering (denoted by $\widetilde{N}$) may be introduced. It turns out that the kinetically normal-ordered current is related to the conventionally normal-ordered

current in a simple fashion [28, 17]

$$\widetilde{N}J(x) = NJ(x) + \frac{e}{2\pi}A_1(x) \tag{15}$$

therefore $\widetilde{N}J$ has the same commutator (14) as NJ. It was proven in [17] that NJ is the consistent current operator and $\widetilde{N}J$ is the covariant current operator. Now it is very easy to compute the consistent and covariant Gauss law commutators. The Gauss law operators are defined as ($\partial_{x^1} \equiv \partial_1$)

$$G(x) = \partial_1 \frac{\delta}{e\delta A_1(x)} - iNJ(x) \tag{16}$$

$$\widetilde{G}(x) = \partial_1 \frac{\delta}{e\delta A_1(x)} - i\widetilde{N}J(x) \tag{17}$$

Here $A_1(x)$ is treated as a function of space only and the time variable x^0 as a parameter, i.e. $(\delta/\delta A_1(x^0, x^1))A_1(x^0, y^1) = \delta(x^1 - y^1)$. The consistent Gauss law commutator is determined by the current commutator (14),

$$[G(x^0, x^1), G(x^0, y^1)] = \frac{i}{2\pi}\delta'(x^1 - y^1) \tag{18}$$

whereas for the covariant case the functional derivatives contribute, as well,

$$[\widetilde{G}(x^0, x^1), \widetilde{G}(x^0, y^1)] = -\frac{i}{2\pi}\delta'(x^1 - y^1). \tag{19}$$

Therefore, the covariant Schwinger term is minus the consistent one, (18). This relative minus sign is precisely as in Table 1. Observe that the covariant current is indeed gauge invariant, $[\widetilde{G}(x^0, x^1), \widetilde{N}J(x^0, y^1)] = 0$, as it must be. In fact, the relative minus sign between the consistent and covariant Schwinger terms is a consequence of this gauge invariance of $\widetilde{N}J$, and therefore independent of all possible conventions.

A generalization of the above results to the nonabelian case is straight forward. The two versions of normal-ordering are defined as in the abelian case, and they lead to the same relation as in (15), up to an additional colour index

$$\widetilde{N}J^a(x) = NJ^a(x) + \frac{e}{2\pi}A_1^a(x). \tag{20}$$

Further, the current commutator acquires a canonical piece as well,

$$[NJ^a(x^0, x^1), NJ^b(x^0, y^1)] = -if^{abc}NJ^c(x^0, x^1)\delta(x^1 - y^1) - \frac{i}{2\pi}\delta^{ab}\delta'(x^1 - y^1) \tag{21}$$

(for the commutator $[\widetilde{N}J^a(x), \widetilde{N}J^b(y)]$, the same expression is obtained, again with NJ^c on the r.h.s., *not* $\widetilde{N}J^c$, as is obvious from (20)). The generator of time-independent gauge transformations on gauge fields,

$$\delta^a(x) := (\delta^{ab}\partial_1 + ef^{acb}A_1^c(x))\frac{\delta}{e\delta A_1^b(x)} \tag{22}$$

obeys the commutation relation

$$[\delta^a(x^0,x^1),\delta^b(x^0,y^1)] = -f^{abc}\delta^c(x^0,x^1)\delta(x^1-y^1). \tag{23}$$

The consistent and covariant Gauss law operators are defined as

$$G^a(x) = \delta^a(x) - iNJ^a(x) \tag{24}$$

and

$$\widetilde{G}^a(x) = \delta^a(x) - i\widetilde{N}J^a(x) \tag{25}$$

respectively. Their anomalous commutators may be easily computed,

$$[G^a(x^0,x^1),G^b(x^0,y^1)] + f^{abc}G^c(x^0,x^1)\delta(x^1-y^1) = \frac{i}{2\pi}\delta^{ab}\delta'(x^1-y^1) \tag{26}$$

$$\begin{aligned}&[\widetilde{G}^a(x^0,x^1),\widetilde{G}^b(x^0,y^1)] + f^{abc}\widetilde{G}^c(x^0,x^1)\delta(x^1-y^1) = \\ &-\frac{i}{2\pi}\delta^{ab}\delta'(x^1-y^1) + \frac{i}{2\pi}f^{abc}A^c(x^0,x^1)\delta(x^1-y^1).\end{aligned} \tag{27}$$

As in the abelian case, the anomalous commutators agree with the ones in Table 1, and again this is most easily seen for the relative minus sign of the $\delta'(x^1-y^1)$ term. This relative sign may be related to the fact that the covariant current has to transform covariantly under a gauge transformation (i.e., the δ' terms must cancel)

$$[\widetilde{G}^a(x^0,x^1),\widetilde{N}J^b(x^0,y^1)] = -f^{abc}\widetilde{N}J^c(x^0,x^1)\delta(x^1-y^1) \tag{28}$$

as may be checked easily.

3.2 Calculation of Hosono and Seo

In this section we shall use the Hosono and Seo approach [18] for the calculation of the equal-time commutators of the covariant and consistent Gauss law operator. The calculation is performed in Minkowski space, $g_{\mu\nu} = \mathrm{diag}\,(1,-1)$, $\varepsilon_{01} = 1$, with the gamma matrices obeying the usual Clifford algebra relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, and $\gamma_5 = \gamma^0\gamma^1$. The anti-Hermitian matrices t^i are the generators of a non-abelian algebra $\left[t^a,t^b\right] = f^{abc}t^c$, and we denote $A_k = A_k^a t^a$.

The Hamiltonian of the chiral fermion interacting with an external gauge potential is

$$\mathcal{H}(A) = -i\int dx\,\left[\bar{\psi}(t,x)\,\gamma^1\frac{1+\gamma_5}{2}\left(\partial_1 + A_1^a(t,x)\,t^a\right)\psi(t,x)\right] \tag{29}$$

where we chose the Weyl gauge ($A^0(t,x) = 0$).

We expand the Fermion field as

$$\psi(t,x) = \sum_n \alpha_n(t)\zeta_n(t,x), \tag{30}$$

where $\zeta_n(t,x)$ are eigenfunctions of the full Hamiltonian (29) with eigenvalues $E_n(t)$. In the quantized theory the α_n are treated as operators satisfying the canonical anticommutation relation

$$\left\{\alpha_n,\alpha_m^+\right\} = \delta_{nm} \tag{31}$$

and the Dirac vacuum is defined as

$$\begin{aligned} \alpha_n(t)\left|0,A(t)\right\rangle_S &= 0, \quad E_n(t)>0, \\ \alpha_n^+(t)\left|0,A(t)\right\rangle_S &= 0, \quad E_n(t)<0. \end{aligned} \tag{32}$$

Observe that the expansion of the Fermion field operators w.r.t. the eigenfunctions of the full Hamiltonian (29) automatically implies that we use the Schroedinger picture.

Singular operator products are regularized in [18] by an exponential damping of high frequencies. The regularized current reads

$$\begin{aligned} \left(j^{\mu a}(x)\right)_{reg} &= \sum_{n,m}\alpha_n^+(t)\,\zeta_n^+(t,x)\,e^{-(\varepsilon/2)E_n^2(t)}\gamma^0\gamma^\mu\frac{1+\gamma_5}{2}e^{-(\varepsilon/2)E_m^2(t)}\zeta_m(t,x)\,\alpha_m(t) \\ &= \sum_{n,m}\alpha_n^+(t)\,\zeta_n^+(t,x)\,e^{-(\varepsilon/2)E_n^2(t)}t^a e_m^{-(\varepsilon/2)E_m^2(t)}\zeta_m(t,x)\,\alpha_m(t) \end{aligned} \tag{33}$$

(where $j^0=j^1$ was used in the second line, which holds for the chiral current (33)). The current in (33) is regularized covariantly, therefore it will lead to the covariant anomaly and Schwinger term. The consistent current J^μ is obtained by adding the Bardeen–Zumino polynomial Δj^μ,

$$J^\mu(x) = j^\mu(x) + \Delta j^\mu(x), \tag{34}$$

$$\Delta j^\mu(x) = -\frac{i}{4\pi}t^a\varepsilon^{\mu\nu}\mathrm{tr}\left(t^a A_\nu\right). \tag{35}$$

These currents lead to the covariant and consistent anomalies

$$\mathcal{A}^a_{cov}(x) = -\left(D^\mu\langle j_\mu\rangle\right)^a(x) = \frac{i}{2\pi}\varepsilon_{\mu\nu}\mathrm{tr}\left(t^a\left(\partial^\mu A^\nu + A^\mu A^\nu\right)\right)(x) \tag{36}$$

and

$$\mathcal{A}^a_{con}(x) = -\left(D^\mu\langle J_\mu\rangle\right)^a(x) = \frac{i}{4\pi}\varepsilon_{\mu\nu}\mathrm{tr}\,t^a\partial^\mu A^\nu(x). \tag{37}$$

The covariant ($\tilde{G}^a$) and consistent (G^a) Gauss law operators read

$$\tilde{G}^a(x) = X^a(x) + j^{0a}(x) \tag{38}$$

$$G^a(x) = X^a(x) + J^{0a}(x) \tag{39}$$

where

$$X^a(x) = -\left(\partial_1\frac{\delta}{\delta A_1^a(x)} + f^{abc}A_1^b(x)\frac{\delta}{\delta A_1^c(x)}\right) \tag{40}$$

generates time-independent gauge transformations of the external gauge field.

Assuming that the non-canonical parts ($n.c.$) of the commutator of the covariant and consistent Gauss laws are c-numbers it is sufficient to consider their vacuum expectation values (VEVs) only. The calculation in the Hosono and Seo approach is rather lengthy, therefore it is performed in the Appendices A – C. Here we just present the final form of the covariant Schwinger term

$$\begin{aligned} \widetilde{ST}^{ab} &= \left\langle \left[\tilde{G}^a(x), \tilde{G}^b(y)\right]_{n.c.} \right\rangle = -\left\langle \left[j^{0a}(x), j^{0b}(y)\right]_{n.c.} \right\rangle = \\ &= \frac{i}{2\pi} \partial_x \delta(x-y) \cdot \mathrm{tr} t^a t^b + \frac{i}{2\pi} \delta(x-y) \cdot \mathrm{tr} t^a \left[A_1(y), t^b\right], \end{aligned} \tag{41}$$

and the consistent one

$$ST^{ab} = \left\langle \left[G^a(x), G^b(y)\right]_{n.c.} \right\rangle = \frac{i}{4\pi} \delta(x-y) \cdot \mathrm{tr}\left(\left[t^a, A_1\right] t^b\right). \tag{42}$$

Comparing the results (41) and (42) with the expressions for the 1+1 dim Schwinger terms in (6) (for the consistent case) and Table 1 (for the covariant case), we find that these terms agree. Therefore, the method of Hosono and Seo reproduces the result of Kelnhofer [14].

4 Method of Wess

In this section we want to review the papers of Schwiebert [20] and Kelnhofer [21] who used the method of Wess [19] for the calculation of the consistent [20] and covariant [21] Schwinger terms (ST), respectively. The central idea of this method is to infer the current commutators from the time derivatives of a (time-ordered) current two-point function, by using the general relation $\partial_x^0 TA(x)B(y) = \delta(x^0 - y^0)[A(x), B(y)]$. As the anomaly is a (covariant) derivative of the current VEV (one-point function), and further current insertions are obtained by functional derivatives w.r.t. the external gauge potential A_a^μ, the current commutator may be related to a functional derivative of the anomaly.

The authors of [20] and [21] used slightly different conventions. For our purposes it is important to have the same conventions for both the consistent and covariant cases, because we want to determine one relative sign. Therefore we shall repeat the major steps in the calculations of [20] and [21] within our specific set of conventions. We choose anti-Hermitean Lie algebra generators λ_a,

$$[\lambda_a, \lambda_b] = f_{abc} \lambda_c \tag{43}$$

where f_{abc} are the structure constants. Further we choose Euclidean conventions in this section ($g^{\mu\nu} = \delta^{\mu\nu}$), mainly because the path integral computation of both the consistent [29] and covariant [30] anomaly was done in Euclidean space as well (for our conventions see e.g. [31]). "Space-time" indices (running from 0 to 1 in $d = 2$ and from 0 to 3 in $d = 4$) are denoted by Greek letters $\mu, \nu, \ldots$ and pure space indices are denoted by latin letters k, l, m. For the Ward operator we choose

$$X_a(x) = -(D_x^\mu)_{ab} \frac{\delta}{\delta A_b^\mu(x)} \equiv -(\delta_{ab} \partial_x^\mu + f_{acb} A_c^\mu(x)) \frac{\delta}{\delta A_b^\mu(x)} \tag{44}$$

$$[X_a(x), X_b(y)] = f_{abc} X_c(x)\delta(x-y). \tag{45}$$

The Euclidean vacuum functional is

$$Z[A] = e^{-W[A]} = \langle 0|T^* e^{-\int dx \widehat{J}^\mu_a(x) A^\mu_a(x)}|0\rangle \tag{46}$$

where A^μ_a is the external gauge potential and $\widehat{J}^\mu_a$ is a covariantly regularized current operator, which necessarily depends on A^μ_a for an anomalous gauge theory. Further T^* is the Lorentz covariantized time-ordered product that results from covariant perturbation theory.

4.1 Consistent case

For the VEV of the consistent current J^μ_a (one-point function) we have ($\int \widehat{J}A \equiv \int dx \widehat{J}^\mu_a(x) A^\mu_a(x)$)

$$\langle 0|T^* J^\mu_a(x) e^{-\int \widehat{J}A}|0\rangle e^W := \frac{\delta W}{\delta A^\mu_a(x)}$$

$$= \langle 0|T^*(\widehat{J}^\mu_a(x) + \int dy \frac{\delta \widehat{J}^\lambda_b(y)}{\delta A^\mu_a(x)} A^\lambda_b(y)) e^{-\int \widehat{J}A}|0\rangle e^W \tag{47}$$

and for the two-point function we get

$$\frac{\delta^2 W}{\delta A^\mu_a(x)\delta A^\nu_b(y)} = -\langle 0|T^* J^\mu_a(x) J^\nu_b(y) e^{-\int \widehat{J}A}|0\rangle e^W$$

$$+ \langle 0|T^* \frac{\delta J^\mu_a(x)}{\delta A^\nu_b(y)} e^{-\int \widehat{J}A}|0\rangle e^W + \frac{\delta W}{\delta A^\mu_a(x)} \frac{\delta W}{\delta A^\nu_b(y)} \tag{48}$$

$$=: -T^{*\mu\nu}_{ab}(x,y) + \Theta^{\mu\nu}_{ab}(y)\delta(x-y) + \cdots \tag{49}$$

where in (49) we have defined abbreviations for the first and second term of (48) and indicated the third (disconnected) term by ellipses. Here it is assumed that J^μ_a depends on A^μ_a only in a local fashion [20].

Now we should re-express the T^* product by the ordinary T product that is defined via θ functions. For the zero- and one-point functions we may simply define the T^* product by the T product, because the latter leads to Lorentz-covariant expressions. On the other hand, for the two-point function $\langle T^* J(x) J(y)\rangle$ there occurs a difference (seagull term $\tau^{\mu\nu}_{ab}$) at coinciding space-time points, and this seagull term is proportional to $\delta(x-y)$ [32, 5]. Denoting the ordinary T product by $T^{\mu\nu}_{ab}(x,y)$, we have

$$T^{*\mu\nu}_{ab}(x,y) = T^{\mu\nu}_{ab}(x,y) + \tau^{\mu\nu}_{ab}(y)\delta(x-y). \tag{50}$$

For the divergence of $T^{\mu\nu}_{ab}$ we get, using the definition of the T product,

$$\partial^\mu_x T^{\mu\nu}_{ab}(x,y) = \partial^\mu_x \Big(\theta(x^0-y^0)\langle 0|(Te^{-\int_{x^0}^\infty \widehat{J}A}) J^\mu_a(x)(Te^{-\int_{y^0}^{x^0} \widehat{J}A}) J^\nu_b(y)(Te^{-\int_{-\infty}^{y^0} \widehat{J}A})|0\rangle$$

$$+((\mu,a,x)\leftrightarrow(\nu,b,y))\Big)e^W$$

$$= \delta(x^0-y^0)\langle 0|T[J_a^0(x),J_b^\nu(y)]e^{-\int \widehat{J}A}|0\rangle e^W + \langle 0|T\partial_x^\mu J_a^\mu(x)J_b^\nu(y)e^{-\int \widehat{J}A}|0\rangle e^W$$
$$- \langle 0|T[J_a^0(x),\int_{z^0=x^0}\mathbf{d}z\widehat{J}_c^\lambda(z)A_c^\lambda(z)]J_b^\nu(y)e^{-\int \widehat{J}A}|0\rangle e^W \tag{51}$$

where $\mathbf{d}z$ is w.r.t. the spacial coordinates only. The term containing $\partial_x^\mu J_a^\mu(x)$ does not produce δ functions and may therefore be neglected. Further, $\widehat{J}$ in the third term may be replaced by J without introducing δ function like contributions. For the commutator we use (in our Euclidean conventions J_b^ν is anti-Hermitean)

$$\delta(x^0-y^0)[J_a^0(x),J_b^\nu(y)] = f_{abc}J_c^\nu(y)\delta(x-y)+C_{ab}^{0\nu}(y)\delta(x-y)+S_{ab}^{0\nu k}(y)\partial_x^k\delta(x-y) \tag{52}$$

Re-inserting this commutator into (51) and omitting disconnected terms we get

$$\partial_x^\mu T_{ab}^{\mu\nu}(x,y) = C_{ab}^{0\nu}(y)\delta(x-y)+S_{ab}^{0\nu k}(y)\partial_x^k\delta(x-y)$$
$$+ f_{abc}\frac{\delta W}{\delta A_c^\nu(y)}\delta(x-y) - f_{adc}A_d^\lambda(x)T_{cb}^{\lambda\nu}(x,y). \tag{53}$$

This result has to be related to the functional derivative of the consistent anomaly, where the consistent anomaly itself is defined as

$$\mathcal{A}_a(x) := X_a(x)W[A]. \tag{54}$$

Explicitly, we have in 2 and 4 dimensions ($A_\mu \equiv A_a^\mu\lambda_a$)

$$d=2:\quad \mathcal{A}_a(x) = c_1\epsilon^{\mu\nu}\mathrm{tr}\,\lambda_a\partial^\mu A^\nu \tag{55}$$

$$d=4,\quad \mathcal{A}_a(x) = c_2\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\lambda_a\partial^\mu(A^\nu\partial^\rho A^\sigma+\frac{1}{2}A^\nu A^\rho A^\sigma) \tag{56}$$

where c_1 and c_2 are some constants. From these explicit expressions we may express the functional derivatives of the anomalies as

$$\frac{\delta\mathcal{A}_a(x)}{\delta A_b^\nu(y)} = I_{ab}^{\mu\nu}(x)\partial_x^\mu\delta(x-y)+(\partial_x^\mu I_{ab}^{\mu\nu}(x))\delta(x-y) = I_{ab}^{\mu\nu}(y)\partial_x^\mu\delta(x-y) \tag{57}$$

where the last equality follows from properties of the δ function. Explicitly we have

$$d=2,\quad I_{ab}^{\mu\nu}(y) = c_1\epsilon^{\mu\nu}\mathrm{tr}\,\lambda_a\lambda_b \tag{58}$$

$$d=4,\quad I_{ab}^{\mu\nu}(y) = \frac{c_2}{2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\Big(\{\lambda_a,\lambda_b\}(2\partial^\rho A^\sigma+A^\rho A^\sigma)-\lambda_a A^\rho\lambda_b A^\sigma\Big). \tag{59}$$

For later convenience we also note that

$$\frac{\delta\mathcal{A}_b(y)}{\delta A_a^\mu(x)} = -I_{ba}^{\nu\mu}(y)\partial_x^\nu\delta(x-y)+(\partial_y^\nu I_{ba}^{\nu\mu}(y))\delta(x-y). \tag{60}$$

On the other hand, we may use the definition (54) of the anomaly (and expression (44) for the Ward operator) to relate the functional derivative (57) to the two-point function (49). We get

$$\frac{\delta \mathcal{A}_a(x)}{\delta A_b^\nu(y)} = -f_{abc}\delta(x-y)\frac{\delta W}{\delta A_c^\nu(x)}$$
$$+(\delta_{ac}\partial_x^\mu + f_{adc}A_d^\mu(x))(T_{cb}^{*\mu\nu}(x,y) - \Theta_{cb}^{\mu\nu}(y)\delta(x-y))$$
$$= C_{ab}^{0\nu}(y)\delta(x-y) + S_{ab}^{0\nu k}(y)\partial_x^k\delta(x-y)$$
$$+ \sigma_{ab}^{\mu\nu}(y)\partial_x^\mu\delta(x-y) + f_{adc}A_d^\mu(y)\sigma_{cb}^{\mu\nu}(y)\delta(x-y) \tag{61}$$

where we introduced

$$\sigma_{ab}^{\mu\nu}(y) := \tau_{ab}^{\mu\nu}(y) - \Theta_{ab}^{\mu\nu}(y). \tag{62}$$

Comparing the coefficients of $\delta(x-y)$, $\partial_x^k\delta(x-y)$ and $\partial_x^0\delta(x-y)$ in (57) and (61) leads to

$$C_{ab}^{0\nu}(y) + f_{adc}A_d^\mu(y)\sigma_{cb}^{\mu\nu}(y) = 0 \tag{63}$$
$$S_{ab}^{0\nu k}(y) + \sigma_{ab}^{k\nu}(y) = I_{ab}^{k\nu}(y) \tag{64}$$
$$\sigma_{ab}^{0\nu}(y) = I_{ab}^{0\nu}(y). \tag{65}$$

For a determination of S_{ab}^{00k} and C_{ab}^{00} we need σ_{ab}^{k0} about which we have no information yet (here we slightly deviate from the calculation of [20] and follow the arguments of [21], but the final result will agree with the result of [20] up to the difference in conventions). For this purpose we compute, analogously to (61),

$$\frac{\delta \mathcal{A}_b(y)}{\delta A_a^\mu(x)} = -f_{bac}\delta(x-y)\frac{\delta W}{\delta A_a^\mu(x)}$$
$$+ (\delta_{bc}\partial_y^\nu + f_{bdc}A_d^\nu(y))(T_{ac}^{*\mu\nu}(x,y) - \Theta_{ac}^{\mu\nu}(y)\delta(x-y)) \tag{66}$$

and use

$$\partial_y^\nu T_{ab}^{\mu\nu}(x,y) = \ldots = -C_{ab}^{\mu 0}(y)\delta(x-y) - S_{ab}^{\mu 0k}(y)\partial_x^k\delta(x-y)$$
$$+ f_{dbc}A_d^\nu(y)T_{ac}^{\mu\nu}(x,y) - f_{abc}\frac{\delta W}{\delta A_c^\mu(y)}\delta(x-y) \tag{67}$$

to arrive at

$$\frac{\delta \mathcal{A}_b(y)}{\delta A_a^\mu(x)} = -C_{ab}^{\mu 0}(y)\delta(x-y) - S_{ab}^{\mu 0k}(y)\partial_x^k\delta(x-y)$$
$$+ \delta(x-y)(\delta_{bc}\partial_y^\nu + f_{bdc}A_d^\nu(y))\sigma_{ac}^{\mu\nu}(y) - \sigma_{ab}^{\mu\nu}(y)\partial_x^\nu\delta(x-y). \tag{68}$$

Comparison of coefficients of (60) and (68) leads to

$$- C_{ab}^{\mu 0}(y) + \partial_y^\nu\sigma_{ab}^{\mu\nu}(y) + f_{bdc}A_d^\nu(y)\sigma_{ac}^{\mu\nu}(y) = \partial_y^\nu I_{ba}^{\nu\mu}(y) \tag{69}$$

$$S_{ab}^{\mu 0k}(y) + \sigma_{ab}^{\mu k}(y) = I_{ba}^{k\mu}(y) \tag{70}$$
$$\sigma_{ab}^{\mu 0}(y) = I_{ba}^{0\mu}(y). \tag{71}$$

Together with (63)–(65) this may be solved for S^{00k}_{ab} and C^{00}_{ab}

$$S^{00k}_{ab}(y) = I^{k0}_{ab}(y) - I^{0k}_{ba}(y) \tag{72}$$

$$C^{00}_{ab}(y) = -f_{adc}A^{\mu}_{d}(y)I^{0\mu}_{bc}(y). \tag{73}$$

In addition we find from (69) and (73) the consistency condition

$$\partial^{\nu}_{y}(I^{0\nu}_{ab}(y) - I^{\nu 0}_{ba}(y)) + A^{\nu}_{d}(y)(f_{adc}I^{0\nu}_{bc}(y) + f_{bdc}I^{0\nu}_{ac}(y)) = 0 \tag{74}$$

which holds for both 2 and 4 dimensions, as may be checked easily. So far we have determined the anomalous $[J^0, J^0]$ commutator, see (52), (72) and (73). We still need the commutator of J^0_a and the Ward operator X_b. As X_b does not contain fermionic degrees, this commutator is equal to the action of X_b on J^0_a,

$$X_b(y)J^0_a(x) \equiv \delta(x^0 - y^0)[X_b(y), J^0_a(x)]. \tag{75}$$

This commutator may be inferred from the relation

$$X_b(y)\frac{\delta W}{\delta A^{\mu}_a(x)} = -(D^{\nu}_{y})_{bc}\frac{\delta}{\delta A^{\nu}_{c}(y)}\langle 0|T J^{\mu}_{a}(x)e^{-\int \tilde{J}A}|0\rangle e^{W}$$

$$= \langle 0|T(X_b(y)J^{\mu}_a(x))e^{-\int \tilde{J}A}|0\rangle e^{W} + (D^{\nu}_{y})_{bc}\langle 0|T J^{\mu}_a(x)J^{\nu}_c(y)e^{-\int \tilde{J}A}|0\rangle e^{W}. \tag{76}$$

Here we used the fact that in the one-point function (47) the T^* product is equal to the T product. It is important to use the T product here, because we want to extract the (Lorentz non-covariant) commutator $[J^0_a, X_b]$ directly, without some covariantizing seagulls. Now we assume that the commutator (75) contains no fermionic degrees of freedom, i.e., it may be extracted from the VEV. Using (49) and (50) for the two-point function we find

$$\delta(x^0 - y^0)[X_b(y), J^{\mu}_a(x)] + (D^{\nu}_{y})_{bc}T^{\mu\nu}_{ac}(x,y) = (D^{\nu}_{y})_{bc}(T^{\mu\nu}_{ac} + \sigma^{\mu\nu}_{ac}(y)\delta(x-y)) \tag{77}$$

or, for $\mu = 0$ and using (65),

$$\delta(x^0 - y^0)[J^0_a(x), X_b(y)] = -(D^{k}_{y})_{bc}(I^{0k}_{ac}(y)\delta(x-y)). \tag{78}$$

Actually, for the Gauss operator we only need the Ward operator restricted to purely spacial gauge transformations. In addition it is preferable to get rid of the time coordinate altogether. Therefore we define a spacial Ward operator

$$\mathbf{X}_a(x) := -\int dx^0 (D^{k}_{x})_{ab}\frac{\delta}{\delta A^{k}_{b}(x)} \tag{79}$$

$$[\mathbf{X}_a(x), \mathbf{X}_b(y)] = f_{abc}\mathbf{X}_c(x)\vec{\delta}(x-y) \tag{80}$$

where $\vec{\delta}(x-y)$ is the spacial δ function. The Gauss operator is

$$G_a(x) = J^0_a(x) + \mathbf{X}_a(x). \tag{81}$$

Using (72), (73) and (78) we find for the anomalous part of the commutator (i.e., the Schwinger term)

$$\mathcal{G}_{ab}(x,y) := [G_a(x), G_b(y)] - f_{abc}G_c(x)\delta(x-y)$$

$$= C^{00}_{ab}(y)\vec{\delta}(x-y) + S^{00k}_{ab}(y)\partial^k_x\vec{\delta}(x-y) + [J^0_a(x), \mathbf{X}_b(y)] + [\mathbf{X}_a(x), J^0_b(y)]$$

$$= -(f_{bdc}A^k_d(y)I^{0k}_{ac}(y) + \partial^k_y I^{0k}_{ab}(y))\vec{\delta}(x-y). \tag{82}$$

Before evaluating this expression explicitly for $d = 2$ and $d = 4$, we want to find the analogous result for the covariant case, following [21].

4.2 Covariant case

The VEV of the covariant current $\widetilde{J}^\mu_a$ is related to the consistent one by the Bardeen–Zumino polynomial Λ^μ_a,

$$\langle 0|T^*\widetilde{J}^\mu_a(x)e^{-\int \widehat{J}A}|0\rangle e^W = \langle 0|T^*J^\mu_a(x)e^{-\int \widehat{J}A}|0\rangle e^W + \Lambda^\mu_a(x). \tag{83}$$

This leads to the covariant anomaly $\widetilde{\mathcal{A}}_a(x)$,

$$\widetilde{\mathcal{A}}_a(x) = -(D^\mu_x)_{ab}\langle 0|T^*\widetilde{J}^\mu_b(x)e^{-\int \widehat{J}A}|0\rangle e^W = \mathcal{A}_a(x) - (D^\mu_x)_{ab}\Lambda^\mu_b(x). \tag{84}$$

Explicitly the covariant anomalies are

$$d = 2\,, \quad \widetilde{\mathcal{A}}_a(x) = 2c_1\epsilon^{\mu\nu}\mathrm{tr}\,\lambda_a(\partial^\mu A^\nu + A^\mu A^\nu) \tag{85}$$

$$d = 4\,, \quad \widetilde{\mathcal{A}}_a(x) = 3c_2\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\lambda_a(\partial^\mu A^\nu + A^\mu A^\nu)(\partial^\rho A^\sigma + A^\rho A^\sigma) \tag{86}$$

where the constants c_1, c_2 are the *same* as in the consistent case, see (55) and (56). The two-point functions are defined analogously to (48)–(50) as

$$\frac{\delta}{\delta A^\nu_b(y)}\langle 0|T^*\widetilde{J}^\mu_a(x)e^{-\int \widehat{J}A}|0\rangle e^W = -\langle 0|T^*\widetilde{J}^\mu_a(x)\widetilde{J}^\nu_b(y)e^{-\int \widehat{J}A}|0\rangle e^W$$

$$+\langle 0|T^*\frac{\delta\widetilde{J}^\mu_a(x)}{\delta A^\nu_b(y)}e^{-\int \widehat{J}A}|0\rangle e^W + \dots$$

$$=: -\widetilde{T}^{*\mu\nu}_{ab}(x,y) + \widetilde{\Theta}^{\mu\nu}_{ab}(y)\delta(x-y) + \dots \tag{87}$$

$$=: -\widetilde{T}^{\mu\nu}_{ab}(x,y) - \widetilde{\sigma}^{\mu\nu}_{ab}(y)\delta(x-y) + \dots \tag{88}$$

where the ellipses denote disconnected terms and all definitions are analogous to the consistent case. Further, the computation of $\partial^\mu_x\widetilde{T}^{\mu\nu}_{ab}(x,y)$ is completely analogous to the consistent case, see (51)–(53). Parametrizing the covariant commutator in an analogous way,

$$\delta(x^0 - y^0)[\widetilde{J}^0_a(x), \widetilde{J}^\nu_b(y)] = f_{abc}\widetilde{J}^\nu_c(x)\delta(x-y) + \widetilde{C}^{0\nu}_{ab}(y)\delta(x-y) + \widetilde{S}^{0\nu k}_{ab}(y)\partial^k_x\delta(x-y) \tag{89}$$

leads to a result analogous to (53),

$$\partial_x^\mu \widetilde{T}_{ab}^{\mu\nu}(x,y) = \widetilde{C}_{ab}^{0\nu}(y)\delta(x-y) + \widetilde{S}_{ab}^{0\nu k}(y)\partial_x^k\delta(x-y)$$
$$+ f_{abc}\delta(x-y)\langle 0|T\widetilde{J}_c^\nu(y)e^{-\int \widehat{J}A}|0\rangle e^W - f_{adc}A_d^\lambda(x)\widetilde{T}_{cb}^{\lambda\nu}(x,y). \tag{90}$$

Again, this should be related to the functional derivative of the (covariant) anomaly. We express this functional derivative as

$$\frac{\delta\widetilde{\mathcal{A}}_a(x)}{\delta A_b^\nu(y)} = \widetilde{I}_{ab}^{\mu\nu}(y)\partial_x^\mu\delta(x-y) + \widetilde{B}_{ab}^\nu(y)\delta(x-y) \tag{91}$$

(we do not display the explicit expressions for $\widetilde{I}$ and $\widetilde{B}$ for $d=2$ or $d=4$, because we do not need them in the sequel).

On the other hand, using the definition of $\widetilde{\mathcal{A}}_a$, relating its functional derivative to the two-point function (88) and inserting (90) for $\partial^\mu\widetilde{T}_{ab}^{\mu\nu}$ leads to

$$\frac{\delta\widetilde{\mathcal{A}}_a(x)}{\delta A_b^\nu(y)} = \widetilde{C}_{ab}^{0\nu}(y)\delta(x-y) + S_{ab}^{0\nu k}(y)\partial_x^k\delta(x-y)$$
$$+ \widetilde{\sigma}_{ab}^{\mu\nu}(y)\partial_x^\mu\delta(x-y) + f_{adc}A_d^\mu(y)\widetilde{\sigma}_{cb}^{\mu\nu}(y)\delta(x-y) \tag{92}$$

and therefore to the equations

$$\widetilde{C}_{ab}^{0\nu}(y) = \widetilde{B}_{ab}^\nu(y) - f_{adc}A_d^\mu(y)\widetilde{\sigma}_{cb}^{\mu\nu}(y) \tag{93}$$
$$\widetilde{S}_{ab}^{0\nu k}(y) = \widetilde{I}_{ab}^{k\nu}(y) - \widetilde{\sigma}_{ab}^{k\nu}(y) \tag{94}$$
$$\widetilde{\sigma}_{ab}^{0\nu}(y) = \widetilde{I}_{ab}^{0\nu}(y). \tag{95}$$

Again we miss information on $\widetilde{\sigma}_{ab}^{k0}(y)$, which we may infer from $(\delta\widetilde{\mathcal{A}}_b(y)/\delta A_a^\mu(x))$. We find

$$\frac{\delta\widetilde{\mathcal{A}}_b(y)}{\delta A_a^\mu(x)} = -f_{bac}\delta(x-y)\langle 0|T^*\widetilde{J}_c^\mu(y)e^{-\int \widehat{J}A}|0\rangle e^W$$
$$-(D_y^\nu)_{bc}\frac{\delta}{\delta A_a^\mu(x)}\langle 0|T^*\widetilde{J}_c^\nu(y)e^{-\int \widehat{J}A}|0\rangle e^W$$
$$= -f_{bac}\delta(x-y)\langle 0|T^*\widetilde{J}_c^\mu(y)e^{-\int \widehat{J}A}|0\rangle e^W$$
$$-(D_y^\nu)_{bc}\Big(\frac{\delta}{\delta A_c^\nu(y)}\langle 0|T^*\widetilde{J}_a^\mu(x)e^{-\int \widehat{J}A}|0\rangle e^W - \mathcal{F}_{ab}^{\mu\nu}(x,y)\Big) \tag{96}$$
$$\mathcal{F}_{ab}^{\mu\nu}(x,y) := \frac{\delta\Lambda_a^\mu(x)}{\delta A_b^\nu(y)} - \frac{\delta\Lambda_b^\nu(y)}{\delta A_a^\mu(x)} \tag{97}$$

where we used relation (83) between consistent and covariant current VEV and the commutativity of functional derivatives (see [21]). Computing $\partial_y^\nu\widetilde{T}_{ab}^{\mu\nu}(x,y)$ as in the consistent case yields

$$\frac{\delta\widetilde{\mathcal{A}}_b(y)}{\delta A_a^\mu(x)} = -\widetilde{C}_{ab}^{\mu 0}(y)\delta(x-y) - \widetilde{S}_{ab}^{\mu 0k}(y)\partial_x^k\delta(x-y)$$

$$+ (D^\nu_y)_{bc}\mathcal{F}^{\mu\nu}_{ac}(x,y) + \delta(x-y)(D^\nu_y)_{bc}\widetilde{\sigma}^{\mu\nu}_{ac}(y) - \widetilde{\sigma}^{\mu\nu}_{ab}(y)\partial^\nu_x\delta(x-y). \quad (98)$$

However, as a consequence of the gauge covariance of the covariant current it holds that

$$\frac{\delta\widetilde{\mathcal{A}}_b(y)}{\delta A^\mu_a(x)} \equiv (D^\nu_y)_{bc}\mathcal{F}^{\mu\nu}_{ac}(x,y) \quad (99)$$

as may be checked explicitly [21]. Therefore, the coefficients in (98) are not directly related to the anomaly and have to obey

$$\widetilde{C}^{\mu 0}_{ab}(y) = (D^\nu_y)_{bc}\widetilde{\sigma}^{\mu\nu}_{ac}(y) \quad (100)$$

$$\widetilde{S}^{\mu 0k}_{ab}(y) = -\widetilde{\sigma}^{\mu k}_{ab}(y) \quad (101)$$

$$\widetilde{\sigma}^{\mu 0}_{ab}(y) = 0 \quad (102)$$

and we find

$$\widetilde{S}^{00k}_{ab}(y) = \widetilde{I}^{k0}_{ab}(y) \quad (103)$$

$$\widetilde{C}^{00}_{ab}(y) = \widetilde{B}^{0}_{ab}(y) \quad (104)$$

and the consistency condition

$$\widetilde{B}^{0}_{ab}(y) = (D^\nu_y)_{bc}\widetilde{I}^{0\nu}_{ac}(y) \quad (105)$$

which holds indeed, as may be checked by explicit computation [21]. For the anomalous part of the current commutator this leads to

$$\delta(x^0-y^0)[\widetilde{J}^0_a(x),\widetilde{J}^0_b(y)] - f_{abc}\widetilde{J}^0_c(y)\delta(x-y) =$$

$$= \widetilde{C}^{00}_{ab}(y)\delta(x-y) + \widetilde{S}^{00k}_{ab}(y)\partial^k_x\delta(x-y) \equiv \frac{\delta\widetilde{\mathcal{A}}_a(x)}{\delta A^0_b(y)}. \quad (106)$$

Again, we have to calculate the $[X_b,\widetilde{J}^0_a]$ commutators as in the consistent case. However, the result is simply that each such term in the Gauss operator commutator produces a contribution that is equal to minus the above expression (106), see [21]. Therefore we find for the covariant Gauss operator

$$\widetilde{G}_a(x) = \widetilde{J}^0_a(x) + \mathbf{X}_a(x) \quad (107)$$

the Schwinger term

$$\widetilde{\mathcal{G}}_{ab}(x,y) := [\widetilde{G}_a(x),\widetilde{G}_b(y)] - f_{abc}\widetilde{G}_c(x)\vec{\delta}(x-y) = -\int dy^0\frac{\delta\widetilde{\mathcal{A}}_a(x)}{\delta A^0_b(y)} \quad (108)$$

where the dy^0 integration just serves to get rid of the unwanted y^0 dependence (this just kills a $\delta(x^0-y^0)$, because there is no time derivative in the above expression (108)).

4.3 Explicit evaluation

Now we are in a position to explicitly compute the Schwinger terms both for $d = 2$ and $d = 4$. Starting with the $d = 2$ case, we find from (58) and (82) for the consistent ST

$$
\begin{aligned}
\mathcal{G}_{ab}(x,y) &= -c_1 \epsilon^{0k} f_{bdc} A^k_d(y) \mathrm{tr}\, \lambda_a \lambda_c \delta^{(1)}(x-y) \\
&= -c_1 \epsilon^{0k} \mathrm{tr}\, \lambda_a [\lambda_b, A^k] \delta^{(1)}(x-y)
\end{aligned} \tag{109}
$$

($\delta^{(1)}(x-y) := \delta(x^1 - y^1)$) and for the covariant ST, using (85) and (108)

$$
\widetilde{\mathcal{G}}_{ab}(x,y) = -2c_1 \epsilon^{0k} \mathrm{tr}\, \lambda_a (\lambda_b \partial^k_y \delta^{(1)}(x-y) + [\lambda_b, A^k] \delta^{(1)}(x-y)) \tag{110}
$$

where here and in the following functions always depend on y when the coordinate argument is not written down explicitly. In order to compare with the expressions of Section 2, we omit ϵ^{0k} and multiply by $(1/2) dy^k v_a(x) v_b(y)$, in the indicated order. Here dy^k is a one-form, $v_a(x)$ is a ghost, and all these objects *anti-commute*, e.g., $dy^k v_a(x) = -v_a(x) dy^k$. We find ($A(y) := A^k(y) dy^k$, $v(x) := v_a(x) \lambda_a$)

$$
\mathcal{G}(x,y) = -\frac{c_1}{2} \mathrm{tr}\, (v(x)v(y) + v(y)v(x)) A(y) \delta^{(1)}(x-y) \tag{111}
$$

or, after integrating w.r.t. $\int dx^1 dy^1$

$$
\mathcal{G} = -c_1 \int dy \, \mathrm{tr}\, v^2 A. \tag{112}
$$

In the same fashion, we get for $\widetilde{\mathcal{G}}_{ab}(x,y)$ ($d_y := dy^k \partial^k_y$)

$$
\widetilde{\mathcal{G}}(x,y) = -c_1 \mathrm{tr}\, v(x) \Big(-(d_y \delta^{(1)}(x-y)) v(y) + \delta^{(1)}(x-y)(A(y)v(y) + v(y)A(y)) \Big) \tag{113}
$$

and (where a partial integration has to be performed)

$$
\widetilde{\mathcal{G}} = -c_1 \int dy \, \mathrm{tr}\, v(dv + Av + vA) = -c_1 \int dy \, \mathrm{tr}\, vDv. \tag{114}
$$

Comparing with eq. (6) (for the consistent case) and Table 1 (for the covariant case), we find that the relative sign of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ is in precise agreement.

For the case $d = 4$ we find from (59) and (82)

$$
\begin{aligned}
\mathcal{G}_{ab}(x,y) = -\frac{c_2}{2} \epsilon^{0klm} \mathrm{tr}\, \Big(&[\lambda_a, \lambda_b](\partial^k A^l A^m + A^k \partial^l A^m + A^k A^l A^m) \\
&+ (\lambda_b A^k \lambda_a - \lambda_a A^k \lambda_b) \partial^l A^m \Big) \delta^{(3)}(x-y)
\end{aligned} \tag{115}
$$

(each derivative acts only on its immediate right hand neighbour), or, after omitting ϵ^{0lkm} and multiplying by $(1/2) dy^k dy^l dy^m v_a(x) v_b(y)$

$$
\mathcal{G}(x,y) = -\frac{c_2}{4} \mathrm{tr}\, \Big((v(x)v(y) + v(y)v(x))(dAA + AdA + A^3)
$$

$$+ (v(y)Av(x) + v(x)Av(y))dA\Big)\delta^{(3)}(x-y) \tag{116}$$

and upon integration $\int d^3xd^3y$

$$\mathcal{G} = -\frac{c_2}{2}\int dy\,\mathrm{tr}\,(v^2(dAA + AdA + A^3) + vAvdA). \tag{117}$$

For the covariant ST we find from (86) and (108)

$$\begin{aligned}\widetilde{\mathcal{G}}_{ab}(x,y) = &-3c_2\epsilon^{0klm}\mathrm{tr}\,\lambda_a\Big((\lambda_b\partial_y^k\delta^{(3)}(x-y) + (\lambda_bA^k - A^k\lambda_b)\delta^{(3)}(x-y))(\partial^lA^m + A^lA^m) \\ &+ (\partial^lA^m + A^lA^m)(\lambda_b\partial_y^k\delta^{(3)}(x-y) + (\lambda_bA^k - A^k\lambda_b)\delta^{(3)}(x-y))\Big)\end{aligned} \tag{118}$$

and

$$\begin{aligned}\widetilde{\mathcal{G}}(x,y) = &-\frac{3c_2}{2}\mathrm{tr}\,(-v(x))\Big(((d_y\delta^{(3)}(x-y))v(y) - (vA + Av)\delta^{(3)}(x-y))(dA + A^2) \\ &+ (dA + A^2)((d_y\delta^{(3)}(x-y))v(y) - (vA + Av)\delta^{(3)}(x-y))\Big)\end{aligned} \tag{119}$$

$$\begin{aligned}\widetilde{\mathcal{G}} &= -\frac{3c_2}{2}\int dy\,\mathrm{tr}\,v((dv + Av + vA)(dA + A^2) + (dA + A^2)(dv + Av + vA)) \\ &= -\frac{3c_2}{2}\int dy\,\mathrm{tr}\,v(DvF + FDv).\end{aligned} \tag{120}$$

Again, the relative sign of consistent and covariant ST precisely agrees with the one in eq. (6) (consistent case) and Table 1 (covariant case).

Acknowledgement

This work started at the Erwin Schrödinger Institute (ESI) program "Quantization, generalized BRS cohomology and anomalies" and we would like to thank the ESI for kind hospitality. Further, we thank R.A. Bertlmann for helpful discussions. One of us (T.S.) is grateful to R.A. Bertlmann for the possibility to spend some time in the creative atmosphere of the ESI.

Appendix

A The Schwinger terms of the Gauss Laws

We start with the covariant case and we consider only the non-canonical part of the commutator:

$$\begin{aligned}\left[\tilde{G}^a(x), \tilde{G}^b(y)\right]_{n.c.} &= \left[X^a(x) + j^{0a}(x), X^b(y) + j^{0b}(y)\right]_{n.c.} = \\ &= \left[X^a(x), X^b(y)\right]_{n.c.} + \left[X^a(x), j^{0b}(y)\right]_{n.c.} + \\ &\quad + \left[j^{0a}(x), X^b(y)\right]_{n.c.} + \left[j^{0a}(x), j^{0b}(y)\right]_{n.c.}.\end{aligned} \tag{121}$$

The gauge field is an external field, therefore the commutator

$$\left[X^{a}\left(x\right),X^{b}\left(y\right)\right]_{n.c.} \tag{122}$$

is zero. For the VEV of the commutator

$$\left[j^{0a}\left(x\right),j^{0b}\left(y\right)\right]_{n.c.} \tag{123}$$

we get, after some manipulations,

$$\begin{aligned}\left\langle\left[j^{0a}\left(x\right),j^{0b}\left(y\right)\right]_{n.c.}\right\rangle &= \mathrm{tr}e^{-(\varepsilon/2)\Delta_y}P_{-}\left(t,y,x\right)t^{a}\left[e^{-(\varepsilon/2)\Delta_x}-1\right]\delta\left(x-y\right)t^{b}- \\ &\quad -\left(x,a\leftrightarrow y,b\right),\end{aligned} \tag{124}$$

where $P_{-}\left(t,x,y\right)$ denotes the projection operator (see Appendix C)

$$P_{-}\left(t,x,y\right)=\sum_{E_n<0}\zeta_n^{+}\left(t,x\right)\zeta_n\left(t,x\right). \tag{125}$$

Then (124) gives

$$\mathrm{tr}e^{-\varepsilon\Delta_y}P_{-}^{(0)}\left(y,x\right)t^{a}\left[e^{-\varepsilon\Delta_x}-1\right]\delta\left(x-y\right)t^{b}-\left(x,a\leftrightarrow y,b\right)=$$

$$\begin{aligned}&= \alpha\ \mathrm{tr}\int dE\theta\left(-E\right)e^{-\varepsilon\Delta_y}e^{-iE(x-y)}t^{a}\times \\ &\qquad \times\int dq\left[e^{-\varepsilon\Delta_x}-1\right]e^{-iq(x-y)}t^{b}-\left(x,a\leftrightarrow y,b\right)= \\ &= \alpha\,\mathrm{tr}\int dEdq\theta\left(-E\right)e^{-\varepsilon\Delta_y}e^{-iE(x-y)}t^{a}\left[e^{-\varepsilon\Delta_x}-1\right]e^{-iq(x-y)}t^{b}- \\ &\quad -\left(x,a\leftrightarrow y,b\right)= \\ &= \alpha\,\mathrm{tr}\int dEdq\theta\left(-E\right)e^{-\varepsilon E^2}e^{-iE(x-y)}\left(1-2i\varepsilon EA\left(y\right)\right)t^{a}\times \\ &\qquad \times\left[\left(1+2i\varepsilon qA\left(x\right)\right)e^{-\varepsilon q^2}-1\right]e^{-iq(x-y)}t^{b}-\left(x,a\leftrightarrow y,b\right)=\end{aligned}$$

$$\begin{aligned}&= \alpha\int dEdq\theta\left(-E\right)e^{-\varepsilon E^2}e^{-i(E+q)(x-y)}\left[e^{-\varepsilon q^2}-1\right]\cdot\mathrm{tr}t^{a}t^{b}-\left(x,a\leftrightarrow y,b\right)- \\ &\quad -i\alpha\int dEdq\theta\left(-E\right)e^{-\varepsilon E^2}\left[e^{-\varepsilon q^2}-1\right]e^{-i(E+q)(x-y)}2\varepsilon E\cdot\mathrm{tr}A\left(y\right)t^{a}t^{b}+ \\ &\quad +\left(x,a\leftrightarrow y,b\right)+ \\ &\quad +i\alpha\int dEdq\theta\left(-E\right)e^{-\varepsilon E^2}2\varepsilon qe^{-\varepsilon q^2}\cdot\mathrm{tr}t^{a}A\left(x\right)t^{b}-\left(x,a\leftrightarrow y,b\right)=\end{aligned}$$

$$\begin{aligned}&= \alpha\int d\xi e^{-i\xi(x-y)}\int dE\theta\left(-E\right)e^{-\varepsilon E^2}\left[e^{-\varepsilon(\xi-E)^2}-1\right]\cdot\mathrm{tr}t^{a}t^{b}-\left(x,a\leftrightarrow y,b\right)- \\ &\quad -i\alpha\int d\xi e^{-i\xi(x-y)}\int dE\theta\left(-E\right)2\varepsilon Ee^{-\varepsilon E^2}\left[e^{-\varepsilon(\xi-E)^2}-1\right]\cdot\mathrm{tr}A\left(y\right)t^{a}t^{b}+ \\ &\quad +\left(x,a\leftrightarrow y,b\right)+\end{aligned}$$

$$
\begin{aligned}
& + i\alpha \int d\xi e^{-i\xi(x-y)} \int dE \theta(-E)\, 2\varepsilon q e^{-\varepsilon E^2} e^{-\varepsilon(\xi-E)^2} \cdot \mathrm{tr} t^a A(x)\, t^b - \\
& - (x, a \leftrightarrow y, b) = \\
= \quad & \alpha \int d\xi e^{-i\xi(x-y)} \int dE e^{-\varepsilon E^2} \left[e^{-\varepsilon(\xi-E)^2} - 1 \right] (\theta(-E) - \theta(E)) \cdot \mathrm{tr} t^a t^b - \\
& - i\alpha \left\{ \left[\int d\xi e^{-i\xi(x-y)} \int dE \theta(-E)\, 2\varepsilon E e^{-\varepsilon E^2} \left[e^{-\varepsilon(\xi-E)^2} - 1 \right] - \right.\right. \\
& \left. - \int d\xi e^{-i\xi(x-y)} \int dE \theta(E)\, 2\varepsilon(\xi - E)\, e^{-\varepsilon E^2} e^{-\varepsilon(\xi-E)^2} \right] \cdot \mathrm{tr} A(y)\, t^a t^b - \\
& \left. - (x, a \leftrightarrow y, b) \right\} \overset{\varepsilon \to 0}{\sim} \\
\sim \quad & - \alpha \int d\xi e^{-i\xi(x-y)} \int dE e^{-\varepsilon E^2} \left[e^{-\varepsilon E^2} \left(1 - \varepsilon \xi^2 + 2\varepsilon E \xi \right) - 1 \right] \varepsilon(E) \cdot \mathrm{tr} t^a t^b - \\
& - i\alpha \left\{ \int d\xi e^{-i\xi(x-y)} \times \right. \\
& \qquad \times \left[\int dE 2\varepsilon E e^{-\varepsilon E^2} e^{-\varepsilon(\xi-E)^2} (\theta(-E) + \theta(E)) - \right. \qquad \qquad \setminus \overset{\varepsilon \to 0}{\to} 0 \\
& \qquad\quad - \int dE \theta(-E)\, 2\varepsilon E e^{-\varepsilon E^2} - \qquad \qquad \qquad \qquad \setminus \overset{\varepsilon \to 0}{\to} 1 \\
& \qquad\quad \left. - \int dE \theta(E)\, 2\varepsilon \xi e^{-\varepsilon E^2} e^{-\varepsilon(\xi-E)^2} \right] \times \qquad \qquad \setminus \overset{\varepsilon \to 0}{\to} 0 \\
& \qquad \left. \times \mathrm{tr} A(y)\, t^a t^b - (x, a \leftrightarrow y, b) \right\} \sim \\
\sim \quad & - \alpha \int d\xi \xi e^{-i\xi(x-y)} \int dE e^{-2\varepsilon E^2} 2\varepsilon E \varepsilon(E) \cdot \mathrm{tr} t^a t^b - \\
& - i\alpha \left\{ \int d\xi e^{-i\xi(x-y)} \cdot \mathrm{tr} A(y)\, t^a t^b - \int d\xi e^{-i\xi(x-y)} \cdot \mathrm{tr} A(x)\, t^b t^a \right\} \overset{\varepsilon \to 0}{\equiv} \\
= \quad & - \alpha \int d\xi e^{-i\xi(x-y)} \xi \cdot \mathrm{tr} t^a t^b + 2\pi i \alpha\, \delta(x-y) \cdot \mathrm{tr} t^a \left[A(y), t^b \right] = \\
= \quad & - \frac{i}{2\pi} \partial_x \delta(x-y) \cdot \mathrm{tr} t^a t^b + \frac{i}{2\pi} \delta(x-y) \cdot \mathrm{tr} t^a \left[A(y), t^b \right],
\end{aligned} \tag{126}
$$

where $\alpha = 1/(2\pi)^2$. Then

$$
\left\langle \left[j^{0a}(x), j^{0b}(y) \right]_{n.c.} \right\rangle = - \frac{i}{2\pi} \partial_x \delta(x-y) \cdot \mathrm{tr} t^a t^b + \frac{i}{2\pi} \delta(x-y) \cdot \mathrm{tr} t^a \left[A(y), t^b \right] \tag{127}
$$

and the covariant Schwinger term of the commutator of the full Gauss law operators has the form

$$
\begin{aligned}
\widetilde{ST}^{ab} &= \left\langle \left[\tilde{G}^a(x), \tilde{G}^b(y) \right]_{n.c.} \right\rangle = - \left\langle \left[j^{0a}(x), j^{0b}(y) \right]_{n.c.} \right\rangle = \\
&= \frac{i}{2\pi} \partial_x \delta(x-y) \cdot \mathrm{tr} t^a t^b + \frac{i}{2\pi} \delta(x-y) \cdot \mathrm{tr} t^a \left[A_1(y), t^b \right],
\end{aligned} \tag{128}
$$

where we used the result for the cross-term

$$\left\langle \left[X^a (x), j^{0b} (y) \right] \right\rangle = \frac{i}{2\pi} \left(\partial_1^x \delta^{ac} + f^{aec} A_1^e (x) \right) \delta (x-y) \cdot \mathrm{tr} t^b t^c \tag{129}$$

obtained in Appendix B.

For the commutator of the consistent Gauss laws we get

$$\begin{aligned} & \left[G^a (x), G^b (y) \right] = \\ = \; & \left[\tilde{G}^a (x), \tilde{G}^b (y) \right] + \left[\tilde{G}^a (x), \Delta j^{0b} (y) \right] + \left[\Delta j^{0a} (x), \tilde{G}^b (y) \right] = \\ = \; & f^{abc} \tilde{G}^c (x) \delta (x-y) = \\ = \; & f^{abc} G^c (x) \delta (x-y) - f^{abc} \Delta j^{0c} (y) \delta (x-y), \end{aligned} \tag{130}$$

where we used the equality

$$\widetilde{ST}^{ab} + \left[\tilde{G}^a (x), \Delta j^{0b} (y) \right] + \left[\Delta j^{0a} (x), \tilde{G}^b (y) \right] = 0. \tag{131}$$

which results from

$$\begin{aligned} \left[\tilde{G}^a (x), \Delta j^{0b} (y) \right] &= \left[X^a (x), \Delta j^{0b} (y) \right] = X^a (x) \Delta j^{0b} (y) = \\ &= \frac{i}{4\pi} \varepsilon^{01} \left(\delta^{ac} \partial^\mu + f^{aec} A^{\mu e} (x) \right) \frac{\delta}{\delta A^{\mu c} (x)} \mathrm{tr} \left(t^b A_1 (y) \right) = \\ &= \frac{i}{4\pi} \left(\delta^{ac} \partial_x^1 + f^{aec} A^{1e} (x) \right) \delta (x-y) \cdot \mathrm{tr} t^b t^c. \end{aligned} \tag{132}$$

Therefore

$$\begin{aligned} ST^{ab} &= -f^{abc} \Delta j^{0c} (y) \delta (x-y) = \\ &= \frac{i}{4\pi} \varepsilon^{0\nu} \mathrm{tr} \left(f^{abc} t^c A_\nu \right) \delta (x-y) = \\ &= \frac{i}{4\pi} \delta (x-y) \cdot \mathrm{tr} \left(\left[t^a, A_1 \right] t^b \right). \end{aligned} \tag{133}$$

B The cross-term

For the VEV of the cross-term

$$\left[X^a (x), j^{0b} (y) \right] \tag{134}$$

we get

$$\begin{aligned} \left\langle \left[X^a (x), j^{0b} (y) \right] \right\rangle &= -\left\langle \left(\partial_x^1 \delta^{ac} + f^{aec} A^{1e} (x) \right) \frac{\delta}{\delta A^{1c} (x)} j^{0b} (y) \right\rangle = \\ &= -\left(\partial_x^1 \delta^{ac} + f^{aec} A^{1e} (x) \right) \left\langle \frac{\delta}{\delta A^{1c} (x)} j^{0b} (y) \right\rangle. \end{aligned} \tag{135}$$

Because

$$
\begin{aligned}
& \left(\frac{\delta}{\delta A^{1c}(x)} e^{-(\varepsilon/2)\Delta_y}\right) P_-(y,z) \overset{\varepsilon\to 0}{\sim} \\
\sim\ & \left(\frac{\delta}{\delta A^{1c}(x)} e^{-(\varepsilon/2)\Delta_y}\right) P_-^{(0)}(y,z) = \\
=\ & \frac{1}{2\pi}\left(\frac{\delta}{\delta A^{1c}(x)} e^{-(\varepsilon/2)\Delta_y}\right)\int dE\theta(-E)\, e^{iE(y-z)} \overset{\varepsilon\to 0}{=} \\
=\ & -\frac{i}{2\pi}\int dE\theta(-E)\,\varepsilon E e^{iE(y-z)} e^{-(\varepsilon/2)p^2}\delta(x-y)\, t^c
\end{aligned} \tag{136}
$$

and

$$
\begin{aligned}
& -\frac{i}{2\pi}\int dE\theta(-E)\,\varepsilon E e^{iE(y-z)} e^{-(\varepsilon/2)E^2} e^{-(\varepsilon/2)\overleftarrow{\Delta}_z} \\
=\ & -\frac{i}{2\pi}\int dE\theta(-E)\,\varepsilon E e^{iE(y-z)} e^{-\varepsilon E^2},
\end{aligned} \tag{137}
$$

we obtain

$$
\begin{aligned}
& \left\langle \frac{\delta}{\delta A^{1c}(x)} j^{0b}(y)\right\rangle = \\
=\ & \lim_{z\to y}\operatorname{tr} t^b\left[\left(\frac{\delta}{\delta A^{1c}(x)} e^{-(\varepsilon/2)\Delta_y}\right) P_-(y,z) e^{-(\varepsilon/2)\overleftarrow{\Delta}_z} + \right. \\
& \left. + e^{-(\varepsilon/2)\Delta_y} P_-(y,z)\left(\frac{\delta}{\delta A^{1c}(x)} e^{-(\varepsilon/2)\overleftarrow{\Delta}_z}\right)\right] = \\
=\ & -\frac{i}{\pi}\int \theta(-E)\,\varepsilon E e^{-\varepsilon E^2} dE\,\delta(x-y)\cdot \operatorname{tr} t^b t^c.
\end{aligned} \tag{138}
$$

The integral is

$$
\int \theta(-E)\,\varepsilon E e^{-\varepsilon E^2} dE = -\frac{1}{2} \tag{139}
$$

and therefore

$$
\left\langle \frac{\delta}{\delta A^{1c}(x)} j^{0b}(y)\right\rangle = \frac{i}{2\pi}\,\delta(x-y)\cdot \operatorname{tr} t^b t^c. \tag{140}
$$

So, for the commutator

$$
\left\langle \left[X^a(x), j^{0b}(y)\right]\right\rangle
$$

we finally get

$$
\left\langle \left[X^a(x), j^{0b}(y)\right]\right\rangle = \frac{i}{2\pi}\left(\partial_1^x \delta^{ac} + f^{aec} A_1^e(x)\right)\,\delta(x-y)\cdot \operatorname{tr} t^b t^c. \tag{141}
$$

C The projection operator

For our purposes we expand the projection operator (125)

$$
\begin{aligned}
P_{-}(t,x,y) &= \langle x| \oint_{C_-} \frac{dE}{2\pi i} \frac{1}{E-H(t)} |y\rangle = \\
&= \langle x| \oint_{C_-} \frac{dE}{2\pi i} \frac{1}{E-H_0-V(t)} |y\rangle = \\
&= \langle x| \oint_{C_-} \frac{dE}{2\pi i} \frac{1}{E-H_0} |y\rangle + \\
&\quad + \langle x| \oint_{C_-} \frac{dE}{2\pi i} \frac{1}{E-H_0} V(t) \frac{1}{E-H_0} |y\rangle + \ldots = \\
&= P_{-}^{(0)}(x,y) + P_{-}^{(1)}(t,x,y) + \ldots, \qquad (142)
\end{aligned}
$$

where C_- is a contour surrounding the negative real axis in the complex E plane.

For the calculation of the commutators it is sufficient to consider only the first term of (142)

$$
\begin{aligned}
P_{-}^{(0)}(x,y) &= \sum_{E_n<0} \zeta_n(x)\,\zeta_n^{+}(y) = \qquad (143) \\
&= \frac{1}{2\pi} \int dE\theta(-E)\, e^{iE(x-y)}. \qquad (144)
\end{aligned}
$$

References

[1] S. Adler, Phys. Rev. 177 (1969), 2426.

[2] J. S. Bell and R. Jackiw, Nuovo Cim. A60 (1969), 47.

[3] W. A. Bardeen, Phys. Rev. 184 (1969), 1848.

[4] R. Jackiw and K. Johnson, Phys. Rev. 182 (1969), 1459.

[5] R. Jackiw, in: “Current Algebras and Anomalies”, eds. S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten, World Scientific, Singapore 1985.

[6] L. D. Faddeev, Phys. Lett. B145 (1984), 81.

[7] J. Mickelsson, Lett. Math. Phys. 7 (1983), 45; Commun. Math. Phys. 97 (1985), 361.

[8] L.D. Faddeev and S.L. Shatashvili, Theor. Math. Phys. 60 (1985), 770.

[9] L.D. Faddeev and S.L. Shatashvili, Phys. Lett. B167 (1986), 225.

[10] J. Wess and B. Zumino, Phys. Lett. B37 (1971), 95.

[11] R. Stora, in: "Recent Progress in Gauge Theories", ed. H. Lehmann, 1983 Cargese Lectures, Nato ASI Series, Plenum Press, NY 1984.

[12] B. Zumino, in: "Relativity, Groups and Topology 2", eds. B. S. DeWitt and R. Stora, 1983 Les Houches Lectures, North Holland, Amsterdam 1984.

[13] I. Tsutsui, Phys. Lett. B229 (1989), 51.

[14] G. Kelnhofer, J. Math. Phys. 34 (1993), 3901.

[15] C. Ekstrand, Ph.D. thesis, Royal Institute of Technology, Stockholm, April 1999.

[16] C. Ekstrand, to appear in J. Math. Phys., hep-th/9903147.

[17] C. Adam, Ann. Phys. 265 (1998), 218.

[18] S. Hosono and K. Seo, Phys. Rev. D38 (1988), 1296.

[19] J. Wess, in: 5th Adriatic Meeting in Particle Physics (Dubrovnik 1986), eds. M. Martinis and I. Andric, World Scientific, Singapore 1987.

[20] C. Schwiebert, Phys. Lett. B241 (1990), 223.

[21] G. Kelnhofer, Z. Phys. C52 (1991), 89.

[22] J. M. Bismut and D. S. Freed, Commun. Math. Phys. 106 (1986), 159; Commun. Math. Phys. 107 (1986), 103.

[23] M. F. Atiyah and I. M. Singer, Proc. Nat. Acad. Sci. 81 (1984), 2596.

[24] C. Ekstrand, hep-th/0002238.

[25] A. L. Carey, J. Mickelsson and M. K. Murray, Commun. Math. Phys. 183 (1997), 707.

[26] B. Zumino, Nucl. Phys. B253 (1985), 477.

[27] C. Adam, Phys. Lett. B455 (1999), 197.

[28] K. Isler, C. Schmid and C. A. Trugenberger, Nucl. Phys. B301 (1988), 327.

[29] L. Alvarez-Gaume and P. Ginsparg, Nucl. Phys. B243 (1984), 449; Ann. Phys. 161 (1985), 423.

[30] K. Fujikawa, Phys. Rev. D29 (1984), 285; Phys. Rev. D31 (1985), 341.

[31] R. Bertlmann, "Anomalies in Quantum Field Theory", Clarendon Press, Oxford 1996.

[32] D. Gross and R. Jackiw, Nucl. Phys. B14 (1969), 269.